3.3 Rules for Differentiation

What you will learn about . . .

- Positive Integer Powers, Multiples, Sums, and Differences
- Products and Quotients
- Negative Integer Powers of x
- Second and Higher Order Derivatives

and why . . .

These rules help us find derivatives of functions analytically more efficiently.

Positive Integer Powers, Multiples, Sums, and Differences

The first rule of differentiation is that the derivative of every constant function is the zero function.

RULE 1 Derivative of a Constant Function

If f is the function with the constant value c, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Proof of Rule 1 If $f(x) = c$ is a function with a constant value c, then

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0. \qquad \blacksquare$$

The next rule is a first step toward a rule for differentiating any polynomial.

RULE 2 Power Rule for Positive Integer Powers of x

If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof of Rule 2 If $f(x) = x^n$, then

$$\frac{d}{dx}(x^n) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}.$$

We can expand $(x + h)^n$ using the Binomial Theorem as follows:

$$\begin{aligned}(x+h)^n &= x^n + n\cdot x^{n-1}h + \frac{n(n-1)}{2}\cdot x^{n-2}h^2 + \cdots + n\cdot xh^{n-1} + h^n \\ &= x^n + n\cdot x^{n-1}h + h^2\cdot[\text{stuff}]\end{aligned}$$

Notice that after the first two terms of the expansion, every term has a factor of h to a power greater than or equal to 2. This means we can factor h^2 out of all the remaining terms (leaving a polynomial expression we have chosen to call "stuff" since its particulars become irrelevant). Now we can complete the proof.

$$\begin{aligned}\frac{d}{dx}(x^n) &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{x^n + n\cdot x^{n-1}h + h^2\cdot[\text{stuff}] - x^n}{h} \\ &= \lim_{h\to 0} \frac{n\cdot x^{n-1}h + h^2\cdot[\text{stuff}]}{h} \\ &= \lim_{h\to 0} (n\cdot x^{n-1} + h\cdot[\text{stuff}]) \\ &= nx^{n-1} + 0\cdot[\text{stuff}] \\ &= nx^{n-1} \qquad \blacksquare\end{aligned}$$

The Power Rule says: To differentiate x^n, multiply by n and subtract 1 from the exponent. For example, the derivatives of x^2, x^3, and x^4 are $2x^1$, $3x^2$, and $4x^3$, respectively.

RULE 3 The Constant Multiple Rule

If u is a differentiable function of x and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Proof of Rule 3

$$\begin{aligned}\frac{d}{dx}(cu) &= \lim_{h\to 0} \frac{cu(x+h) - cu(x)}{h} \\ &= c \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \\ &= c\frac{du}{dx} \qquad \blacksquare\end{aligned}$$

Rule 3 says that if a differentiable function is multiplied by a constant, then its derivative is multiplied by the same constant. Combined with Rule 2, it enables us to find the derivative of any monomial quickly; for example, the derivative of $7x^4$ is $7(4x^3) = 28x^3$.

To find the derivatives of polynomials, we need to be able to differentiate sums and differences of monomials. We can accomplish this by applying the Sum and Difference Rule.

Denoting Functions by u and v

The functions we work with when we need a differentiation formula are likely to be denoted by letters like f and g. When we apply the formula, we do not want to find the formula using these same letters in some other way. To guard against this, we denote the functions in differentiation rules by letters like u and v that are not likely to be already in use.

RULE 4 The Sum and Difference Rule

If u and v are differentiable functions of x, then their sum and difference are differentiable at every point where u and v are differentiable. At such points,

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}.$$

Proof of Rule 4

We use the difference quotient for $f(x) = u(x) + v(x)$.

$$\begin{aligned}\frac{d}{dx}[u(x) + v(x)] &= \lim_{h\to 0} \frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h} \\ &= \lim_{h\to 0}\left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h}\right] \\ &= \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} \\ &= \frac{du}{dx} + \frac{dv}{dx}\end{aligned}$$

The proof of the rule for the difference of two functions is similar. $\blacksquare$

EXAMPLE 1 Differentiating a Polynomial

Find $\dfrac{dp}{dt}$ if $p = t^3 + 6t^2 - \dfrac{5}{3}t + 16$.

SOLUTION

By Rule 4 we can differentiate the polynomial term-by-term, applying Rules 1 through 3 as we go.

$$\begin{aligned}\frac{dp}{dt} &= \frac{d}{dt}(t^3) + \frac{d}{dt}(6t^2) - \frac{d}{dt}\left(\frac{5}{3}t\right) + \frac{d}{dt}(16) && \text{Sum and Difference Rule}\\ &= 3t^2 + 6\cdot 2t - \frac{5}{3} + 0 && \text{Constant and Power Rules}\\ &= 3t^2 + 12t - \frac{5}{3}\end{aligned}$$

Now Try Exercise 5.

EXAMPLE 2 Finding Horizontal Tangents

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

SOLUTION

The horizontal tangents, if any, occur where the slope dy/dx is zero. To find these points, we

(a) calculate dy/dx:

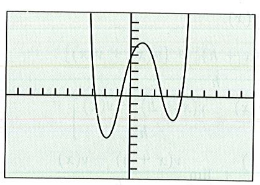
$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x.$$

(b) solve the equation $dy/dx = 0$ for x:

$$\begin{aligned}4x^3 - 4x &= 0\\ 4x(x^2 - 1) &= 0\\ x &= 0, 1, -1.\end{aligned}$$

The curve has horizontal tangents at $x = 0$, 1, and -1. The corresponding points on the curve (found from the equation $y = x^4 - 2x^2 + 2$) are $(0, 2)$, $(1, 1)$, and $(-1, 1)$. You might wish to graph the curve to see where the horizontal tangents go.

Now Try Exercise 7.



[–10, 10] by [–10, 10]

Figure 3.18 The graph of $y = 0.2x^4 - 0.7x^3 - 2x^2 + 5x + 4$ has three horizontal tangents. (Example 3)

The derivative in Example 2 was easily factored, making an algebraic solution of the equation $dy/dx = 0$ correspondingly simple. When a simple algebraic solution is not possible, the solutions to $dy/dx = 0$ can still be found to a high degree of accuracy by using the SOLVE capability of your calculator.

EXAMPLE 3 Using Calculus and Calculator

As can be seen in the viewing window $[-10, 10]$ by $[-10, 10]$, the graph of $y = 0.2x^4 - 0.7x^3 - 2x^2 + 5x + 4$ has three horizontal tangents (Figure 3.18). At what points do these horizontal tangents occur?

continued

On Rounding Calculator Values

Notice in Example 3 that we rounded the x-values to four significant digits when we presented the answers. The calculator actually presented many more digits, but there was no practical reason for writing all of them. When we used the calculator to compute the corresponding y-values, however, we *used the x-values stored in the calculator*, not the rounded values. We then rounded the y-values to four significant digits when we presented the ordered pairs. Significant "round-off errors" can accumulate in a problem if you use rounded intermediate values for doing additional computations, so avoid rounding until the final answer.

Formula Tip

You can remember the Product Rule with the phrase "the first times the derivative of the second plus the second times the derivative of the first."

SOLUTION

First we find the derivative

$$\frac{dy}{dx} = 0.8x^3 - 2.1x^2 - 4x + 5.$$

Using the calculator solver, we find that $0.8x^3 - 2.1x^2 - 4x + 5 = 0$ when $x \approx -1.862$, 0.9484, and 3.539. We use the calculator again to evaluate the original function at these x-values and find the corresponding points to be approximately $(-1.862, -5.321)$, $(0.9484, 6.508)$, and $(3.539, -3.008)$.

Now Try Exercise 11.

Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives and the derivative of the difference of two functions is the difference of their derivatives, the derivative of the product of two functions is *not* the product of their derivatives.

For instance,

$$\frac{d}{dx}(x\cdot x) = \frac{d}{dx}(x^2) = 2x, \quad \text{while} \quad \frac{d}{dx}(x)\cdot\frac{d}{dx}(x) = 1\cdot 1 = 1.$$

The derivative of a product is actually the sum of *two* products, as we now explain.

RULE 5 The Product Rule

The product of two differentiable functions u and v is differentiable, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Proof of Rule 5 We begin, as usual, by applying the definition.

$$\frac{d}{dx}(uv) = \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

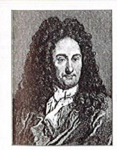
To change the fraction into an equivalent one that contains difference quotients for the derivatives of u and v, we subtract and add $u(x+h)v(x)$ in the numerator. Then,

$$\begin{aligned}\frac{d}{dx}(uv) &= \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}\\ &= \lim_{h\to 0}\left[u(x+h)\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h}\right] && \text{Factor and separate.}\\ &= \lim_{h\to 0}u(x+h)\cdot\lim_{h\to 0}\frac{v(x+h) - v(x)}{h} + v(x)\cdot\lim_{h\to 0}\frac{u(x+h) - u(x)}{h}.\end{aligned}$$

As h approaches 0, $u(x+h)$ approaches $u(x)$ because u, being differentiable at x, is continuous at x. The two fractions approach the values of dv/dx and du/dx, respectively, at x. Therefore

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Gottfried Wilhelm Leibniz (1646–1716)



The method of limits used in this book was not discovered until nearly a century after Newton and Leibniz, the discoverers of calculus, had died.

To Leibniz, the key idea was the *differential*, an infinitely small quantity that was almost like zero, but which—unlike zero—could be used in the denominator of a fraction. Thus, Leibniz thought of the derivative dy/dx as the quotient of two differentials, dy and dx.

The problem was explaining why these differentials sometimes became zero and sometimes did not! See Exercise 59.

Some 17th-century mathematicians were confident that the calculus of Newton and Leibniz would eventually be found to be fatally flawed because of these mysterious quantities. It was only after later generations of mathematicians had found better ways to prove their results that the calculus of Newton and Leibniz was accepted by the entire scientific community.

EXAMPLE 4 Differentiating a Product

Find $f'(x)$ if $f(x) = (x^2 + 1)(x^3 + 3)$.

SOLUTION

From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\begin{aligned} f'(x) &= \frac{d}{dx}[(x^2+1)(x^3+3)] = (x^2+1)(3x^2) + (x^3+3)(2x) \\ &= 3x^4 + 3x^2 + 2x^4 + 6x \\ &= 5x^4 + 3x^2 + 6x. \end{aligned}$$

Now Try Exercise 13.

We could also have done Example 4 by multiplying out the original expression and then differentiating the resulting polynomial. That alternate strategy will not work, however, on a product like $x^2 \sin x$.

Just as the derivative of the product of two differentiable functions is not the product of their derivatives, the derivative of a quotient of two functions is not the quotient of their derivatives. What happens instead is this:

Formula Tip

Since order is important in subtraction, be sure to set up the numerator of the Quotient Rule correctly:

v times the derivative of u

minus

u times the derivative of v.

You can remember the Quotient Rule with the phrase "bottom times the derivative of the top minus the top times the derivative of the bottom, all over the bottom squared."

RULE 6 The Quotient Rule

At a point where $v \neq 0$, the quotient $y = u/v$ of two differentiable functions is differentiable, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Proof of Rule 6

$$\begin{aligned} \frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h\to 0} \frac{\dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x)}}{h} \\ &= \lim_{h\to 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{hv(x+h)v(x)} \end{aligned}$$

To change the last fraction into an equivalent one that contains the difference quotients for the derivatives of u and v, we subtract and add $v(x)u(x)$ in the numerator. This allows us to continue with

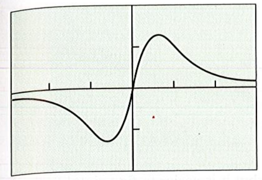
$$\begin{aligned} \frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h\to 0} \frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{hv(x+h)v(x)} \\ &= \lim_{h\to 0} \frac{v(x)\dfrac{u(x+h)-u(x)}{h} - u(x)\dfrac{v(x+h)-v(x)}{h}}{v(x+h)v(x)}. \end{aligned}$$

Taking the limits in both the numerator and denominator now gives us the Quotient Rule. ■

EXAMPLE 5 Supporting Computations Graphically

Differentiate $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$. Support graphically.

continued



[–3, 3] by [–2, 2]

Figure 3.19 The graph of

$$y = \frac{4x}{(x^2+1)^2}$$

and the graph of

$$y = \text{NDER}\left(\frac{x^2-1}{x^2+1}, x\right)$$

appear to be the same. (Example 5)

SOLUTION

We apply the Quotient Rule with $u = x^2 - 1$ and $v = x^2 + 1$:

$$\begin{aligned} f'(x) &= \frac{(x^2+1)\cdot 2x - (x^2-1)\cdot 2x}{(x^2+1)^2} \qquad \frac{v(du/dx) - u(dv/dx)}{v^2} \\ &= \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2+1)^2} \\ &= \frac{4x}{(x^2+1)^2}. \end{aligned}$$

The graphs of $y_1 = f'(x)$ calculated above and of $y_2 = \text{NDER}\,(f(x), x)$ are shown in Figure 3.19. The fact that they appear to be identical provides strong graphical support that our calculations are indeed correct. *Now Try Exercise 19.*

EXAMPLE 6 Working with Numerical Values

Let $y = uv$ be the product of the functions u and v. Find $y'(2)$ if

$$u(2) = 3, \quad u'(2) = -4, \quad v(2) = 1, \quad \text{and} \quad v'(2) = 2.$$

SOLUTION

From the Product Rule, $y' = (uv)' = uv' + vu'$. In particular,

$$\begin{aligned} y'(2) &= u(2)v'(2) + v(2)u'(2) \\ &= (3)(2) + (1)(-4) \\ &= 2. \end{aligned}$$

Now Try Exercise 23.

Negative Integer Powers of x

The rule for differentiating negative powers of x is the same as Rule 2 for differentiating positive powers of x, although our proof of Rule 2 does not work for negative values of n. We can now extend the Power Rule to negative integer powers by a clever use of the Quotient Rule.

RULE 7 Power Rule for Negative Integer Powers of x

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof of Rule 7 If n is a negative integer, then $n = -m$, where m is a positive integer. Hence, $x^n = x^{-m} = 1/x^m$, and

$$\begin{aligned} \frac{d}{dx}(x^n) = \frac{d}{dx}\left(\frac{1}{x^m}\right) &= \frac{x^m \cdot \dfrac{d}{dx}(1) - 1 \cdot \dfrac{d}{dx}(x^m)}{(x^m)^2} \\ &= \frac{0 - mx^{m-1}}{x^{2m}} \\ &= -mx^{-m-1} \\ &= nx^{n-1}. \end{aligned}$$

■

EXAMPLE 7 Using the Power Rule

Find an equation for the line tangent to the curve

$$y = \frac{x^2 + 3}{2x}$$

at the point $(1, 2)$. Support your answer graphically.

SOLUTION

We could find the derivative by the Quotient Rule, but it is easier to first simplify the function as a sum of two powers of x.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x^2}{2x} + \frac{3}{2x}\right)\\ &= \frac{d}{dx}\left(\frac{1}{2}x + \frac{3}{2}x^{-1}\right)\\ &= \frac{1}{2} - \frac{3}{2}x^{-2}\end{aligned}$$

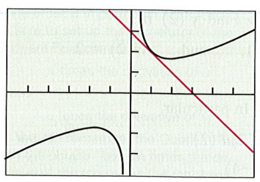
The slope at $x = 1$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = \left[\frac{1}{2} - \frac{3}{2}x^{-2}\right]_{x=1} = \frac{1}{2} - \frac{3}{2} = -1.$$

The line through $(1, 2)$ with slope $m = -1$ is

$$\begin{aligned}y - 2 &= (-1)(x - 1)\\ y &= -x + 1 + 2\\ y &= -x + 3.\end{aligned}$$

We graph $y = (x^2 + 3)/2x$ and $y = -x + 3$ (Figure 3.20), observing that the line appears to be tangent to the curve at $(1, 2)$. Thus, we have graphical support that our computations are correct. *Now Try Exercise 27.*



$[-6, 6]$ by $[-4, 4]$

Figure 3.20 The line $y = -x + 3$ appears to be tangent to the graph of

$$y = \frac{x^2 + 3}{2x}$$

at the point $(1, 2)$. (Example 7)

Second and Higher Order Derivatives

The derivative $y' = dy/dx$ is called the *first derivative* of y with respect to x. The first derivative may itself be a differentiable function of x. If so, its derivative,

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2},$$

is called the *second derivative* of y with respect to x. If y'' ("y double-prime") is differentiable, its derivative,

$$y''' = \frac{dy''}{dx} = \frac{d^3y}{dx^3},$$

is called the *third derivative* of y with respect to x. The names continue as you might expect they would, except that the multiple-prime notation begins to lose its usefulness after about three primes. We use

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} \quad \text{"y super n"}$$

to denote the **nth derivative** of y with respect to x. (We also use d^ny/dx^n.) Do not confuse $y^{(n)}$ with the nth power of y, which is y^n.

Technology Tip

HIGHER ORDER DERIVATIVES WITH NDER

Some graphers will allow the *nesting* of the NDER function,

NDER (NDER($f(x)$, x), a),

but such nesting, in general, is safe only to the second derivative. Beyond that, the error buildup in the algorithm makes the results unreliable.

EXAMPLE 8 Finding Higher Order Derivatives

Find the first four derivatives of $y = x^3 - 5x^2 + 2$.

SOLUTION

The first four derivatives are:

First derivative:	$y' = 3x^2 - 10x$;
Second derivative:	$y'' = 6x - 10$;
Third derivative:	$y''' = 6$;
Fourth derivative:	$y^{(4)} = 0$.

This function has derivatives of all orders, the fourth and higher order derivatives all being zero. *Now Try Exercise 33.*

EXAMPLE 9 Finding Instantaneous Rate of Change

An orange farmer currently has 200 trees yielding an average of 15 bushels of oranges per tree. She is expanding her farm at the rate of 15 trees per year, while improving husbandry is improving her average annual yield by 1.2 bushels per tree. What is the current (instantaneous) rate of increase of her total annual production of oranges?

SOLUTION

Let the functions t and y be defined as follows.

$$\begin{aligned}t(x) &= \text{the number of trees } x \text{ years from now.}\\ y(x) &= \text{yield per tree } x \text{ years from now.}\end{aligned}$$

Then $p(x) = t(x)y(x)$ is the total production of oranges in year x. We know the following values.

$$t(0) = 200, \quad y(0) = 15$$
$$t'(0) = 15, \quad y'(0) = 1.2$$

We need to find $p'(0)$, where $p = ty$.

$$\begin{aligned}p'(0) &= t(0)y'(0) + y(0)t'(0)\\ &= (200)(1.2) + (15)(15)\\ &= 465\end{aligned}$$

The rate we seek is 465 bushels per year. *Now Try Exercise 51.*

Quick Review 3.3 (For help, go to Sections 1.2 and 3.1.)

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–6, write the expression as a sum of powers of x.

1. $(x^2 - 2)(x^{-1} + 1)$
2. $\left(\frac{x}{x^2 + 1}\right)^{-1}$
3. $3x^2 - \frac{2}{x} + \frac{5}{x^2}$
4. $\frac{3x^4 - 2x^3 + 4}{2x^2}$
5. $(x^{-1} + 2)(x^{-2} + 1)$
6. $\frac{x^{-1} + x^{-2}}{x^{-3}}$
7. Find the positive roots of the equation
$$2x^3 - 5x^2 - 2x + 6 = 0$$
and evaluate the function $y = 500x^6$ at each root. Round your answers to the nearest integer, but only in the final step.

8. If $f(x) = 7$ for all real numbers x, find

(a) $f(10)$. (b) $f(0)$.

(c) $f(x + h)$. (d) $\lim_{x\to 0} \frac{f(x) - f(a)}{x - a}$.

9. Find the derivatives of these functions with respect to x.

(a) $f(x) = \pi$ (b) $f(x) = \pi^2$ (c) $f(x) = \pi^{15}$

10. Find the derivatives of these functions with respect to x using the definition of the derivative.

(a) $f(x) = \frac{x}{\pi}$ (b) $f(x) = \frac{\pi}{x}$

Section 3.3 Exercises

In Exercises 1–6, find dy/dx.

1. $y = -x^2 + 3$ **2.** $y = \frac{x^3}{3} - x$

3. $y = 2x + 1$ **4.** $y = x^2 + x + 1$

5. $y = \frac{x^3}{3} + \frac{x^2}{2} + x$ **6.** $y = 1 - x + x^2 - x^3$

In Exercises 7–12, find the horizontal tangents of the curve.

7. $y = x^3 - 2x^2 + x + 1$ **8.** $y = x^3 - 4x^2 + x + 2$

9. $y = x^4 - 4x^2 + 1$ **10.** $y = 4x^3 - 6x^2 - 1$

11. $y = 5x^3 - 3x^5$ **12.** $y = x^4 - 7x^3 + 2x^2 + 15$

13. Let $y = (x + 1)(x^2 + 1)$. Find dy/dx **(a)** by applying the Product Rule, and **(b)** by multiplying the factors first and then differentiating.

14. Let $y = (x^2 + 3)/x$. Find dy/dx **(a)** by using the Quotient Rule, and **(b)** by first dividing the terms in the numerator by the denominator and then differentiating.

In Exercises 15–22, find dy/dx. (You can support your answer graphically.)

15. $(x^3 + x + 1)(x^4 + x^2 + 1)$ **16.** $(x^2 + 1)(x^3 + 1)$

17. $y = \frac{2x + 5}{3x - 2}$ **18.** $y = \frac{x^2 + 5x - 1}{x^2}$

19. $y = \frac{(x - 1)(x^2 + x + 1)}{x^3}$ **20.** $y = (1 - x)(1 + x^2)^{-1}$

21. $y = \frac{x^2}{1 - x^3}$ **22.** $y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

23. Suppose u and v are functions of x that are differentiable at $x = 0$, and that $u(0) = 5$, $u'(0) = -3$, $v(0) = -1$, $v'(0) = 2$. Find the values of the following derivatives at $x = 0$.

(a) $\frac{d}{dx}(uv)$ (b) $\frac{d}{dx}\left(\frac{u}{v}\right)$

(c) $\frac{d}{dx}\left(\frac{v}{u}\right)$ (d) $\frac{d}{dx}(7v - 2u)$

24. Suppose u and v are functions of x that are differentiable at $x = 2$ and that $u(2) = 3$, $u'(2) = -4$, $v(2) = 1$, and $v'(2) = 2$. Find the values of the following derivatives at $x = 2$.

(a) $\frac{d}{dx}(uv)$ (b) $\frac{d}{dx}\left(\frac{u}{v}\right)$

(c) $\frac{d}{dx}\left(\frac{v}{u}\right)$ (d) $\frac{d}{dx}(3u - 2v + 2uv)$

25. Which of the following numbers is the slope of the line tangent to the curve $y = x^2 + 5x$ at $x = 3$?

i. 24 ii. $-5/2$ iii. 11 iv. 8

26. Which of the following numbers is the slope of the line $3x - 2y + 12 = 0$?

i. 6 ii. 3 iii. 3/2 iv. 2/3

In Exercises 27 and 28, find an equation for the line tangent to the curve at the given point.

27. $y = \frac{x^3 + 1}{2x}$, $x = 1$ **28.** $y = \frac{x^4 + 2}{x^2}$, $x = -1$

In Exercises 29–32, find dy/dx.

29. $y = 4x^{-2} - 8x + 1$

30. $y = \frac{x^{-4}}{4} - \frac{x^{-3}}{3} + \frac{x^{-2}}{2} - x^{-1} + 3$

31. $y = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$ **32.** $y = 2\sqrt{x} - \frac{1}{\sqrt{x}}$

In Exercises 33–36, find the first four derivatives of the function.

33. $y = x^4 + x^3 - 2x^2 + x - 5$ **34.** $y = x^2 + x + 3$

35. $y = x^{-1} + x^2$ **36.** $y = \frac{x + 1}{x}$

37. Find an equation of the line perpendicular to the tangent to the curve $y = x^3 - 3x + 1$ at the point $(2, 3)$.

38. Find the tangents to the curve $y = x^3 + x$ at the points where the slope is 4. What is the smallest slope of the curve? At what value of x does the curve have this slope?

39. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the x-axis.

40. Find the x- and y-intercepts of the line that is tangent to the curve $y = x^3$ at the point $(-2, -8)$.

41. Find the tangents to *Newton's serpentine,*

$$y = \frac{4x}{x^2 + 1},$$

at the origin and the point $(1, 2)$.

42. Find the tangent to the *witch of Agnesi,*

$$y = \frac{8}{4 + x^2},$$

at the point $(2, 1)$.

43. Use the definition of derivative (given in Section 3.1, Equation 1) to show that

(a) $\frac{d}{dx}(x) = 1$.

(b) $\frac{d}{dx}(-u) = -\frac{du}{dx}$.

44. Use the Product Rule to show that

$$\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx} f(x)$$

for any constant c.

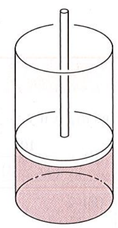
45. Devise a rule for $\frac{d}{dx}\left(\frac{1}{f(x)}\right)$.

When we work with functions of a single variable in mathematics, we often call the independent variable x and the dependent variable y. Applied fields use many different letters, however. Here are some examples.

46. ***Cylinder Pressure*** If gas in a cylinder is maintained at a constant temperature T, the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2},$$

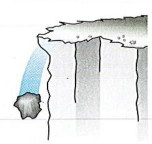
in which a, b, n, and R are constants. Find dP/dV.



47. ***Free Fall*** When a rock falls from rest near the surface of the earth, the distance it covers during the first few seconds is given by the equation

$$s = 4.9t^2.$$

In this equation, s is the distance in meters and t is the elapsed time in seconds. Find ds/dt and d^2s/dt^2.



Group Activity In Exercises 48–52, *work in groups of two or three* to solve the problems.

48. ***The Body's Reaction to Medicine*** The reaction of the body to a dose of medicine can often be represented by an equation of the form

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right),$$

where C is a positive constant and M is the amount of medicine absorbed in the blood. If the reaction is a change in blood pressure, R is measured in millimeters of mercury. If the reaction is a change in temperature, R is measured in degrees, and so on.

Find dR/dM. This derivative, as a function of M, is called the sensitivity of the body to medicine. In Chapter 5, we shall see how to find the amount of medicine to which the body is most sensitive. *Source: Some Mathematical Models in Biology,* Revised Edition, December 1967, PB-202 364, p. 221; distributed by N.T.I.S., U.S. Department of Commerce.

49. **Writing to Learn** Recall that the area A of a circle with radius r is πr^2 and that the circumference C is $2\pi r$. Notice that $dA/dr = C$. Explain in terms of geometry why the instantaneous rate of change of the area with respect to the radius should equal the circumference.

50. **Writing to Learn** Recall that the volume V of a sphere of radius r is $(4/3)\pi r^3$ and that the surface area A is $4\pi r^2$. Notice that $dV/dr = A$. Explain in terms of geometry why the instantaneous rate of change of the volume with respect to the radius should equal the surface area.

51. ***Orchard Farming*** An apple farmer currently has 156 trees yielding an average of 12 bushels of apples per tree. He is expanding his farm at a rate of 13 trees per year, while improved husbandry is boosting his average annual yield by 1.5 bushels per tree. What is the current (instantaneous) rate of increase of his total annual production of apples? Answer in appropriate units of measure.

52. ***Picnic Pavilion Rental*** The members of the Blue Boar society always divide the pavilion rental fee for their picnics equally among the members. Currently there are 65 members and the pavilion rents for $250. The pavilion cost is increasing at a rate of $10 per year, while the Blue Boar membership is increasing at a rate of 6 members per year. What is the current (instantaneous) rate of change in each member's share of the pavilion rental fee? Answer in appropriate units of measure.

Standardized Test Questions

53. **True or False** $\frac{d}{dx}(\pi^3) = 3\pi^2$. Justify your answer.

54. **True or False** The graph of $f(x) = 1/x$ has no horizontal tangents. Justify your answer.

55. Multiple Choice Let $y = uv$ be the product of the functions u and v. Find $y'(1)$ if $u(1) = 2$, $u'(1) = 3$, $v(1) = -1$, and $v'(1) = 1$.

(A) -4 (B) -1 (C) 1 (D) 4 (E) 7

56. Multiple Choice Let $f(x) = x - \frac{1}{x}$. Find $f''(x)$.

(A) $1 + \frac{1}{x^2}$ (B) $1 - \frac{1}{x^2}$ (C) $\frac{2}{x^3}$

(D) $-\frac{2}{x^3}$ (E) does not exist

57. Multiple Choice Which of the following is $\frac{d}{dx}\left(\frac{x+1}{x-1}\right)$?

(A) $\frac{2}{(x-1)^2}$ (B) 0 (C) $-\frac{x^2+1}{x^2}$

(D) $2x - \frac{1}{x^2} - 1$ (E) $-\frac{2}{(x-1)^2}$

58. Multiple Choice Assume $f(x) = (x^2 - 1)(x^2 + 1)$. Which of the following gives the number of horizontal tangents of f?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Extending the Ideas

59. *Leibniz's Proof of the Product Rule* Here's how Leibniz explained the Product Rule in a letter to his colleague John Wallis:

It is useful to consider quantities infinitely small such that when their ratio is sought, they may not be considered zero, but which are rejected as often as they occur with quantities incomparably greater. Thus if we have $x + dx$, dx is rejected. Similarly we cannot have xdx and $dxdx$ standing together, as xdx is incomparably greater than $dxdx$. Hence if we are to differentiate uv, we write

$$\begin{aligned} d(uv) &= (u + du)(v + dv) - uv \\ &= uv + vdu + udv + dudv - uv \\ &= vdu + udv. \end{aligned}$$

Answer the following questions about Leibniz's proof.

(a) What does Leibniz mean by a quantity being "rejected"?

(b) What happened to $dudv$ in the last step of Leibniz's proof?

(c) Divide both sides of Leibniz's formula

$$d(uv) = vdu + udv$$

by the differential dx. What formula results?

(d) Why would the critics of Leibniz's time have objected to dividing both sides of the equation by dx?

(e) Leibniz had a similar simple (but not-so-clean) proof of the Quotient Rule. Can you reconstruct it?

Quick Quiz for AP* Preparation: Sections 3.1–3.3

1. Multiple Choice Let $f(x) = |x + 1|$. Which of the following statements about f are true?

I. f is continuous at $x = -1$.
II. f is differentiable at $x = -1$.
III. f has a corner at $x = -1$.

(A) I only (B) II only (C) III only

(D) I and III only (E) I and II only

2. Multiple Choice If the line normal to the graph of f at the point $(1, 2)$ passes through the point $(-1, 1)$, then which of the following gives the value of $f'(1)$ = ?

(A) -2 (B) 2 (C) $-1/2$ (D) $1/2$ (E) 3

3. Multiple Choice Find dy/dx if $y = \frac{4x-3}{2x+1}$.

(A) $\frac{10}{(4x-3)^2}$ (B) $-\frac{10}{(4x-3)^2}$ (C) $\frac{10}{(2x+1)^2}$

(D) $-\frac{10}{(2x+1)^2}$ (E) 2

4. Free Response Let $f(x) = x^4 - 4x^2$.

(a) Find all the points where f has horizontal tangents.

(b) Find an equation of the tangent line at $x = 1$.

(c) Find an equation of the normal line at $x = 1$.

3.4 Velocity and Other Rates of Change

What you will learn about . . .

- Instantaneous Rates of Change
- Motion Along a Line
- Sensitivity to Change
- Derivatives in Economics

and why . . .

Derivatives give the rates at which things change in the world.

Instantaneous Rates of Change

In this section we examine some applications in which derivatives as functions are used to represent the rates at which things change in the world around us. It is natural to think of change as change with respect to time, but other variables can be treated in the same way. For example, a physician may want to know how change in dosage affects the body's response to a drug. An economist may want to study how the cost of producing steel varies with the number of tons produced.

If we interpret the difference quotient

$$\frac{f(x+h) - f(x)}{h}$$

as the average rate of change of the function f over the interval from x to $x + h$, we can interpret its limit as h approaches 0 to be the rate at which f is changing at the point x.

DEFINITION Instantaneous Rate of Change

The **(instantaneous) rate of change** of f with respect to x at a is the derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

It is conventional to use the word *instantaneous* even when x does not represent time. The word, however, is frequently omitted in practice. When we say *rate of change*, we mean *instantaneous rate of change*.

EXAMPLE 1 Enlarging Circles

(a) Find the rate of change of the area A of a circle with respect to its radius r.

(b) Evaluate the rate of change of A at $r = 5$ and at $r = 10$.

(c) If r is measured in inches and A is measured in square inches, what units would be appropriate for dA/dr?

SOLUTION

The area of a circle is related to its radius by the equation $A = \pi r^2$.

(a) The (instantaneous) rate of change of A with respect to r is

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = \pi \cdot 2r = 2\pi r.$$

(b) At $r = 5$, the rate is 10π (about 31.4). At $r = 10$, the rate is 20π (about 62.8). Notice that the rate of change gets bigger as r gets bigger. As can be seen in Figure 3.21, the same change in radius brings about a bigger change in area as the circles grow radially away from the center.

(c) The appropriate units for dA/dr are square inches (of area) per inch (of radius).

Now Try Exercise 1.